A branch and prune algorithm for the approximation of non-linear AE-solution sets

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ABSTRACT

Non-linear AE-solution sets are a special case of parametric systems of equations where universally quantified parameters appear first. They allow to model many practical situations. A new branch and prune algorithm dedicated to the approximation of non-linear AE-solution sets is proposed. It is based on a new generalized interval (intervals whose bounds are not constrained to be ordered) parametric Hansen-Sengupta operator. In spite of some restrictions on the form of the AE-solution set which can be approximated, it allows to solve problems which were before out of reach of previous numerical methods. Some promising experimentations are presented.

Categories and Subject Descriptors

G.1 [**Numerical Analysis**]: Miscellaneous

General Terms

Theory, Algorithms

Keywords

AE-solution set, branch and prune, Hansen-Sengupta, generalized intervals.

1. INTRODUCTION

A quantified constraint (QC) is a parametric constraint with quantified parameters, e.g. the parametric constraint $ax = 1$ on the variable $x \in \mathbb{R}$ gives rise to the QC

$$(\exists a \in [2, 4])\,(ax = 1)\,. \qquad (1)$$

QCs allow to model a great variety of practical situations, e.g. in control engineering or biology ([7, 6]). QCs are likely to have null or non-null volume solution sets, e.g. (1) whose solution set is $[0.25, 0.5]$. So inner and outer approximations of QCs solution sets are necessary. QCs involving inequalities have been studied by several authors ([9, 5]). In the meantime, QCs involving one equation or a conjunction of equations but whose existentially quantified parameters are not shared between different equations have also been studied ([12, 8]). The situation where some existentially quantified variables are shared between some equations is not well solved today. Although outer approximation of such QCs solution sets can be done thanks to the previously cited works, their inner approximation is still a difficult problem. The quantifier elimination method ([3]) is a formal method that can help to compute both inner and outer approximations of such QCs. However, it is restricted to very small polynomial systems. No numerical algorithm has been yet proposed to solve this problem.

In this paper, we are interested in a specific kind of QC solution sets called AE-solution sets and defined by

$$\{x \in \mathbf{x} \mid\ (\forall u \in \mathbf{u})\,(\exists v \in \mathbf{v})\,(f(u, v, x) = 0)\}$$

where $f = (f_1, \dots, f_m)^T$ is continuously differentiable and $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)^T$, $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_p)^T$ and $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_m)^T$ are interval vectors. We intend to treat the case where the existentially quantified parameters v_k are shared between the different equations. To this end, we need the restriction that there are the same number of equations ($\dim f$) and of existentially quantified variables ($\dim \mathbf{v}$).

We propose a new bisection algorithm based on a parametric Hansen-Sengupta operator. This operator uses generalized intervals (intervals whose bounds are not constrained to be ordered) to improve the pruning in the presence of non-degenerated domains for universally quantified parameters.

Notations. Following [4], intervals are denoted by boldface letters. Intervals considered here are closed, bounded and nonempty, the set of these intervals being denoted by $\mathbb{IR}$. Interval meet and join operations are denoted respectively by $\mathbf{x} \bigwedge \mathbf{y}$ and $\mathbf{x} \bigvee \mathbf{y}$. The midpoint of $\mathbf{x}$ is denoted by $\operatorname{mid} \mathbf{x}$ and the interior of $\mathbf{x}$ is denoted by $\operatorname{int} \mathbf{x}$. Integral intervals are denoted by $[m..n]$. Let $\mathcal{E} = \{e_1, ..., e_n\}$ be an ordered set of indices, the vector $(\mathbf{x}_{e_1}, ..., \mathbf{x}_{e_n})^T$ is denoted by $\mathbf{x}_{\mathcal{E}}$, so that $(\mathbf{x}_1, \dots, \mathbf{x}_n)^T$ is be denoted by $\mathbf{x}_{[1..n]}$. If no confusion is possible, the usual notation $\mathbf{x}$ will be used instead of $\mathbf{x}_{[1..n]}$.

2. GENERALIZED INTERVALS AND AE-SOLUTION SETS

2.1 Generalized intervals

A generalized interval (also called directed interval) is an interval whose bounds are not constrained to be ordered. The set of generalized intervals is denoted by $\mathbb{KR}$. For

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example, $[-1,1]$ is a proper interval and $[1,-1]$ is an improper one. So, related to a set of reals $\{x \in \mathbb{R} \mid a \le x \le b\}$, one can consider two generalized intervals $[a,b]$ and $[b,a]$. It will be convenient to use the operations $\text{dual}\,[a,b] = [b,a]$ and $\text{pro}\,[a,b] = [\min\{a,b\}, \max\{a,b\}]$ to change the proper/improper quality keeping unchanged the underlying set of reals. The proper generalized intervals are identified to classical intervals, so that $\text{pro}\;\mathbf{x}$ is considered as a set of reals whatever is $\mathbf{x} \in \mathbb{KR}$ (we can write e.g. $2 \in \text{pro}\,[3,1]$).

An inclusion is defined for generalized intervals using the same formal expression as in the context of classical intervals: $\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{y}} \le \underline{\mathbf{x}} \wedge \overline{\mathbf{x}} \le \overline{\mathbf{y}}$. As in the context of classical intervals, the generalized interval meet and join operations are defined respectively as the greatest lower bound and the least upper bound of the inclusion partial order. Their expressions are the same as their classical counterparts, e.g. $[2,3] \bigwedge [4,5] = [4,3]$.

The Kaucher arithmetic (also called directed interval arithmetic) extends the classical interval arithmetic to generalized intervals. Its definition can be found e.g. in [12]. In the cases of addition and subtraction, the Kaucher arithmetic has the same formal expressions as the classical interval arithmetic, e.g. $[1,2]+[5,3] = [6,5]$. This is also the case for multiplication and division when only positive intervals are involved, e.g. $[1,2] \times [5,4] = [5,8]$ and $[12,14]/[7,3] = [4,2]$.

2.2 AE-solution sets

United solution sets are generalized to AE-solution sets allowing more general quantifications of the parameters. Both an interval domain and a quantifier are now associated to each parameter. The generalized intervals are well suited for the description of this association: given a generalized interval $\mathbf{a} \in \mathbb{KR}$, the domain associated to a is $\text{pro}\;\mathbf{a}$ and while the quantifier associated to a depends on the proper/improper quality of $\mathbf{a}$. This leads to the following definition.

Definition 1. Let $f : \mathbb{R}^p \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a continuous function and $\mathbf{a} \in \mathbb{KR}^p$ and $\mathbf{b} \in \mathbb{KR}^n$. Then $\Sigma(f,\mathbf{a},\mathbf{b})$ is the following subset of $\mathbb{R}^m$

$$\{x \in \mathbb{R}^m \mid (\forall a_{\mathcal{I}} \in \mathbf{a}_{\mathcal{I}})(\forall b_{\mathcal{I}'} \in \text{pro}\;\mathbf{b}_{\mathcal{I}'})(\exists b_{\mathcal{P}'} \in \mathbf{b}_{\mathcal{P}'})(\exists a_{\mathcal{P}} \in \mathbf{a}_{\mathcal{P}}) (f(a,x)=b)\},$$

where $\mathcal{P} = \{k \in [1..p] \mid \mathbf{a}_k \in \mathbb{IR}\}$, $\mathcal{I} = \{k \in [1..p] \mid \mathbf{a}_k \notin \mathbb{IR}\}$, $\mathcal{P}' = \{k \in [1..n] \mid \mathbf{b}_k \in \mathbb{IR}\}$ and $\mathcal{I}' = \{k \in [1..n] \mid \mathbf{b}_k \notin \mathbb{IR}\}$.

Definition 1 is taken from [12] but we have changed the convention which associates the proper/improper quality of $\mathbf{a}_k$ to the quantifier associated to a_k. This new convention leads to homogeneous notations for united solution sets and AE-solution sets. In particular, if $\mathbf{a}$ and $\mathbf{b}$ are proper then $\Sigma(f,\mathbf{a},\mathbf{b})$ is the usual united solution set.

In the case of linear systems, the parameters are arranged in a matrix so that we retrieve the following usual special cases of linear AE-solution sets: given $\mathbf{A} \in \mathbb{IR}^{n\times m}$ and $\mathbf{b} \in \mathbb{IR}^n$, the united solution set is

$$\Sigma(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^m \mid (\exists A \in \mathbf{A})\,(\exists b \in \mathbf{b})\,(Ax=b)\};$$

the tolerable solution set is

$$\Sigma(\text{dual}\;\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^m \mid (\forall A \in \mathbf{A})\,(\exists b \in \mathbf{b})\,(Ax=b)\};$$

the controllable solution set is

$$\Sigma(\mathbf{A},\text{dual}\;\mathbf{b}) = \{x \in \mathbb{R}^m \mid (\forall b \in \mathbf{b})\,(\exists A \in \mathbf{A})\,(Ax=b)\}.$$

As illustrated by Theorem 1 (taken from [12]), generalized intervals are not only a useful language for quantifiers modeling but also constitute an important analysis tool. It is now stated according to our new convention for the association of a quantifier to the quality proper/improper of an interval.

THEOREM 1. *Let* $\mathbf{A} \in \mathbb{KR}^{m\times n}$ *and* $\mathbf{b} \in \mathbb{KR}^m$. *Then* $x \in \Sigma(\mathbf{A},\mathbf{b})$ *if and only if* $(\text{dual}\;\mathbf{A})x \subseteq \mathbf{b}$.

Theorem 1 homogenizes and generalizes the classical characterizations for united, tolerable and controllable solution sets.

2.3 The generalized Gauss-Seidel operator

Outer approximation of linear AE-solution sets can be done using an extension to generalized intervals of the Gauss-Seidel (GS) operator ([11]). Using our conventions, the generalized GS operator $\Gamma(\mathbf{A},\mathbf{b},\mathbf{x})$ has the same expression as its classical counterpart. It is first defined in the case $n=1$: $\Gamma(\mathbf{a},\mathbf{b},\mathbf{x})$ is equal to $(\mathbf{b}/\mathbf{a}) \bigwedge \mathbf{x}$ if $0 \notin \text{pro}\;\mathbf{a}$ and otherwise to $\mathbf{x}$.

Example 1. Let us consider $\mathbf{a} = [5,3]$, $\mathbf{b} = [6,20]$ and $\mathbf{x} = [3,5]$. So that

$$\Sigma(\mathbf{a},\mathbf{b}) = \{x \in \mathbb{R} \mid (\forall a \in [3,5])\,(\exists b \in [6,20])\,(ax=b)\}.$$

In this simple situation, an easy application of Theorem 1 allows to determine $\Sigma(\mathbf{a},\mathbf{b})$: by Theorem 1, $x \in \Sigma(\mathbf{a},\mathbf{b})$ is equivalent to $(\text{dual}\;\mathbf{a})x \subseteq \mathbf{b}$, i.e. $[3,5]x \subseteq [6,20]$. Obviously, x has to be positive and we immediately obtain the following two inequalities: $6 \le 3x$ and $5x \le 20$. We have therefore $\Sigma(\mathbf{a},\mathbf{b}) = [2,4]$. Now, in this simple situation the Kaucher division $\mathbf{b}/\mathbf{a} = [2,4]$ computes the exact solution set. So that the generalized GS operator leads to $\Gamma(\mathbf{a},\mathbf{b},\mathbf{x}) = (\mathbf{b}/\mathbf{a}) \bigwedge \mathbf{x} = [3,4]$ and computes exactly $\Sigma(\mathbf{a},\mathbf{b}) \bigcap \mathbf{x}$.

Now, the generalized GS operator is defined for square interval matrices of dimension n in the following way:

$$\Gamma_i(\mathbf{A},\mathbf{b},\mathbf{x}) = \Gamma(\mathbf{A}_{ii}\;,\; \mathbf{b}_i - \sum_{j\neq i} \mathbf{A}_{ik}\mathbf{x}_k\;,\; \mathbf{x}_i).$$

The generalized GS operator has the same interpretation as its classical counterpart: if $\Gamma(\mathbf{A},\mathbf{b},\mathbf{x})$ is proper then $\Sigma(\mathbf{A},\mathbf{b}) \bigcap \mathbf{x} \subseteq \Gamma(\mathbf{A},\mathbf{b},\mathbf{x})$. Also, if $\Gamma(\mathbf{A},\mathbf{b},\mathbf{x})$ is not proper then $\Sigma(\mathbf{A},\mathbf{b}) \bigcap \mathbf{x} = \emptyset$.

A preconditioning dedicated to outer approximation of linear AE-solution sets has been proposed in [12]: given a nonsingular real matrix $C \in \mathbb{R}^{n\times n}$, we have

$$\Sigma(\mathbf{A},\mathbf{b}) \bigcap \mathbf{x} \quad\subseteq\quad \Gamma(C\mathbf{A}, C\mathbf{b}, \mathbf{x}).$$

This section is ended by the presentation of a generalized GS operator Γ^* dedicated to non-square interval matrices. It is defined by

$$\Gamma^*_i(\mathbf{A},\mathbf{b},\mathbf{x}) = \bigwedge_{k\in[1..m]} \Gamma(\mathbf{A}_{ik}\;,\; \mathbf{b}_i - \sum_{j\neq k} \mathbf{A}_{ik}\mathbf{x}_k\;,\; \mathbf{x}_i).$$

The interpretation of this operator is the same as previously. In contrast to Γ which needs n computations of the one dimensional GS operator, Γ^* needs $n \times m$ computations of the one dimensional GS operator. No preconditioning will be used with Γ^* because the involved interval matrix is not square.

3. A GENERALIZED HANSEN-SENGUPTA OPERATOR

The Hansen-Sengupta (HS) operator, a special case of interval Newton operators, is widely used to approximate the solutions of a $n \times n$-system of equations. One limitation of this operator is that the parameters of the system of equation are supposed to be known without uncertainty. We now propose a parametric version of the HS operator. Thanks to its definition in the context of generalized intervals, this generalized interval parametric HS operator can deal with existentially and universally quantified parameters.

In the following, we consider a continuously differentiable function $f : \mathbb{R}^p \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, some intervals $\mathbf{a} \in \mathbb{KR}^p$, $\mathbf{b} \in \mathbb{KR}^n$, $\mathbf{x} \in \mathbb{IR}^n$, $\tilde{a} \in \text{pro}\ \mathbf{a}$ and $\tilde{x} \in \mathbf{x}$. Furthermore, $\mathbf{J}^a \in \mathbb{IR}^{n\times p}$ and $\mathbf{J}^x \in \mathbb{IR}^{n\times n}$ are interval enclosures of f'_a and f'_x over pro $\mathbf{a}$ and $\mathbf{x}$. It can be proved (the proof is not provided here because of the lack of space but can be found in [1]) that the non-linear AE-solution set $\Sigma(f, \mathbf{a}, \mathbf{b}) \bigcap \mathbf{x}$ is included inside the following translated linear AE-solution set:

$$\tilde{x} + \Sigma\big(\mathbf{J}^x, \mathbf{b} - f(\tilde{a}, \tilde{x}) - \mathbf{J}^a(\mathbf{a} - \tilde{a})\big). \tag{2}$$

If $\mathbf{a}$ is degenerated (so that $\mathbf{a} = \tilde{a}$ and $\mathbf{J}^a(\mathbf{a} - \tilde{a}) = 0$) and $\mathbf{b} = 0$ then (2) corresponds to the classical interval Newton operator (see [2]). The generalized HS operator $H_f(\mathbf{x}, \mathbf{a}, \mathbf{b})$ is obtained applying the generalized GS operator to (2) after a translation of vector $-\tilde{x}$ and a preconditioning:

$$H_f(\mathbf{a}, \mathbf{b}, \mathbf{x}) := \tilde{x} + \Gamma\big(C\mathbf{J}^x\ ,\ C\mathbf{t}\ ,\ \mathbf{x} - \tilde{x}\big),$$

where $\mathbf{t} = \big(\mathbf{b} - f(\tilde{a}, \tilde{x}) - \mathbf{J}^a(\mathbf{a} - \tilde{a})\big)$. The classical HS operator is retrieved when $\mathbf{a} = \tilde{a}$, and $\mathbf{b} = 0$. As a direct consequence of the properties of the generalized GS operator, the generalized HS operator satisfies the following properties: on one hand if $H_f(\mathbf{a}, \mathbf{b}, \mathbf{x})$ is proper then $\Sigma(f, \mathbf{a}, \mathbf{b}) \bigcap \mathbf{x} \subseteq H_f(\mathbf{a}, \mathbf{b}, \mathbf{x})$. On the other hand, if $H_f(\mathbf{a}, \mathbf{b}, \mathbf{x})$ is not proper then $\Sigma(f, \mathbf{a}, \mathbf{b}) \bigcap \mathbf{x} = \emptyset$. In the context of the classical HS operator, the inclusion $H_f(\tilde{a}, 0, \mathbf{x}) \subseteq \mathbf{x}$ allows to prove the existence of a solution to the equation $f(\tilde{a}, x) = 0$. Unlike its classical counterpart, the generalized HS operator does not offer such a proving power in general. However, the following special case dedicated to non-linear united solution sets will be useful.

THEOREM 2. *Using the notations of this section, suppose that* $\mathbf{a}$ *and* $\mathbf{b}$ *are proper. Then* $H_f(\mathbf{a}, \mathbf{b}, \mathbf{x}) \subseteq \text{int}\,\mathbf{x}$ *implies* $\mathbf{x} \subseteq \Sigma(f, \mathbf{a}, \mathbf{b})$*, i.e.* $(\forall a \in \mathbf{a})\,(\forall b \in \mathbf{b})\,(\exists x \in \mathbf{x})\,(f(a, x) = b)$.

PROOF. Let us consider any $\alpha \in \mathbf{a}$ and $\beta \in \mathbf{b}$ and define $g(x) = f(\alpha, x) - \beta$. So we have to prove $(\exists x \in \mathbf{x})\,(g(x) = 0)$. We have $g' = f'_x$ and we can use $\mathbf{J}^x$ as derivative enclosure for the function g. Using the classical mean-value extension of f, we obtain

$$g(\tilde{x}) = f(\alpha, \tilde{x}) - \beta \in f(\tilde{a}, \tilde{x}) - \beta + \mathbf{J}^a(\mathbf{a} - \tilde{a}). \tag{3}$$

Now, the classical HS operator of $g(x)$ is

$$\mathbf{y} := \tilde{x} + \Gamma(C\mathbf{J}^x, -Cg(\tilde{x}), \mathbf{x} - \tilde{x}).$$

By the inclusion monotonicity of the GS operator (see [2]), (3) implies

$$\mathbf{y} \subseteq \tilde{x} + \Gamma\big(C\mathbf{J}^x, C(\ \beta - f(\tilde{a}, \tilde{x}) - \mathbf{J}^a(\mathbf{a} - \tilde{a})\ ), \mathbf{x} - \tilde{x}\big).$$

Once more by inclusion monotonicity, $\beta \in \mathbf{b}$ implies

$$\mathbf{y} \subseteq \tilde{x} + \Gamma\big(C\mathbf{J}^x, C(\ \mathbf{b} - f(\tilde{a}, \tilde{x}) - \mathbf{J}^a(\mathbf{a} - \tilde{a})\ ), \mathbf{x} - \tilde{x}\big).$$

This last expression corresponds to $H_f(\mathbf{a}, \mathbf{b}, \mathbf{x})$. Therefore, using the statement hypothesis, we obtain $\mathbf{y} \subseteq \text{int}\,\mathbf{x}$. Finally, using the proving power of the classical HS operator, we have $(\exists x \in \mathbf{x})\,(g(x) = 0)$, which corresponds to the statement to be proved. □

The proving power provided by Theorem 2 only involves classical intervals but this parametric Hansen-Sengupta operator has neither been proposed nor nor used before up to our knowledge.

Finally, in the case of functions $f : \mathbb{R}^p \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ where $m \neq n$, some pruning can be achieved using Γ^* instead of Γ in the expression of the generalized HS operator.

4. A BISECTION ALGORITHM

In this section, we are interested by the approximation of non-linear AE-solution sets of the following form:

$$\mathbb{S}_f(\mathbf{u}, \mathbf{v}) := \{x \in \mathbb{R}^m \mid (\forall u \in \mathbf{u})\,(\exists v \in \mathbf{v})\,(f(u, v, x) = 0)\}$$

where $f : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ so that there are the same number of equations as existentially quantified parameters. This is the main restriction of the algorithm proposed here. On the other hand, the algorithm proposed here will not bisect the space related to v. As a consequence, the box $\mathbf{v}$ should be small enough for the algorithm to be efficient. From now on, $\mathbf{J}^u$ and $\mathbf{J}^v$ and $\mathbf{J}^x$ will stand for some interval enclosures of respectively f'_u and f'_v and f'_x. Also, the midpoint of an interval $\mathbf{x}$ will be denoted by $\hat{\mathbf{x}}$ so as to obtain more readable expressions. The generalized HS operator is applied in two different ways.

4.1 Pruning $\mathbf{x}$

The generalized HS operator is used to prune a box $\mathbf{x}$ to a new box $\mathbf{x}'$ in the following way:

$$\mathbf{x}' = \hat{\mathbf{x}} + \Gamma^*\big(\mathbf{J}^x, \mathbf{s}, \mathbf{x} - \hat{\mathbf{x}}\big)$$

where $\mathbf{s} = -f(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{x}}) - \mathbf{J}^u(\text{dual}\ \mathbf{u} - \hat{\mathbf{u}}) - \mathbf{J}^v(\mathbf{v} - \hat{\mathbf{v}})$. Notice that dual $\mathbf{u}$ and $\mathbf{v}$ are used. Therefore u is universally quantified while v is existentially quantified. Therefore, by the properties of the generalized HS operator, if $\mathbf{x}'$ is proper then

$$\mathbb{S}_f(\mathbf{u}, \mathbf{v}) \bigcap \mathbf{x}\ =\ \mathbb{S}_f(\mathbf{u}, \mathbf{v}) \bigcap \mathbf{x}'$$

Also, if $\mathbf{x}'$ is not proper then $\mathbb{S}_f(\mathbf{u}, \mathbf{v}) \bigcap \mathbf{x} = \emptyset$.

4.2 Pruning $\mathbf{v}$ and proving $\mathbf{x} \subseteq \mathbb{S}_f(\mathbf{u}, \mathbf{v})$

The generalized HS operator is also used with the inverse midpoint preconditioning ($C = (\text{mid}\ \mathbf{J}^v)^{-1}$) to prune a box $\mathbf{v}$ to a new box $\mathbf{v}'$ in the following way:

$$\mathbf{v}' = \hat{\mathbf{v}} + \Gamma\big(C\mathbf{J}^v, C\mathbf{t}, \mathbf{v} - \hat{\mathbf{v}}\big)$$

where $\mathbf{t} = -f(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{x}}) - \mathbf{J}^u(\mathbf{u} - \hat{\mathbf{u}}) - \mathbf{J}^x(\mathbf{x} - \hat{\mathbf{x}})$. Notice that only proper intervals are involved so that the following united solution set is actually approximated:

$$\{v \in \mathbf{v} \mid (\exists u \in \mathbf{u})\,(\exists x \in \mathbf{x})\,(f(u, v, x) = 0)\}\,.$$

This computation has several purposes. On one hand, it prunes the values of $\mathbf{v}$ which are not useful for any $u \in \mathbf{u}$ and $x \in \mathbf{x}$: if $\mathbf{v}'$ is proper then

$$\mathbb{S}_f(\mathbf{u}, \mathbf{v}) \bigcap \mathbf{x}\ =\ \mathbb{S}_f(\mathbf{u}, \mathbf{v}') \bigcap \mathbf{x}.$$

Also, if $\mathbf{v}'$ is not proper then $\mathbb{S}_f(\mathbf{u},\mathbf{v}) \bigcap \mathbf{x} = \emptyset$. On the other hand, it allows to use Theorem 2 for an inner test: if $\mathbf{v}'$ is proper and $\mathbf{v}' \subseteq \operatorname{int} \mathbf{v}$ then $\mathbf{x} \subseteq \mathbb{S}_f(\mathbf{u},\mathbf{v})$.

4.3 The algorithm

Our branch and prune algorithm bisects both $\mathbf{x}$ and $\mathbf{u}$. The pseudo-code hereinafter presented describes a simpler version that only bisects $\mathbf{x}$. The main data structure of the algorithm is a heap of boxes $(\mathbf{u},\mathbf{v},\mathbf{x})$ denoted by $\mathcal{H}$. It contains the boxes which have not yet been proved to be inside nor outside $\mathbb{S}_f(\mathbf{u},\mathbf{v})$. It is sorted by decreasing width, so that the top box corresponds to the biggest interval width. Also, a list $\mathcal{L}$ contains the boxes which are proved to be inside $\mathbb{S}_f(\mathbf{u},\mathbf{v})$. The algorithm picks a box in $\mathcal{H}$, applies the generalized HS operator as described in Section 4.1 and Section 4.2 and bisects the box (a simple midpoint bisection is used) if it was not proved to be inside nor outside $\mathbb{S}_f(\mathbf{u},\mathbf{v})$. The algorithm is stopped when the largest width of the boxes $\mathbf{x}$ in $\mathcal{H}$ is less than a given ω. As a consequence

1 **while** $\texttt{WidthMax}(\mathcal{H}) > \omega$ **do**
2 $\quad(\mathbf{u},\mathbf{v},\mathbf{x}) \leftarrow \texttt{Pop}(\mathcal{H})$;
3 $\quad$Refresh derivatives enclosures;
4 $\quad\mathbf{t} \leftarrow -f(\hat{\mathbf{u}},\hat{\mathbf{v}},\hat{\mathbf{x}}) - \mathbf{J}^u(\mathbf{u}-\hat{\mathbf{u}}) - \mathbf{J}^x(\mathbf{x}-\hat{\mathbf{x}})$;
5 $\quad C \leftarrow (\operatorname{mid} \mathbf{J}^v)^{-1}$;
6 $\quad\mathbf{v}' \leftarrow \hat{\mathbf{v}} + \Gamma\big(C\mathbf{J}^v, C\mathbf{t}, \mathbf{v}-\hat{\mathbf{v}}\big)$;
7 $\quad$**if** $\texttt{IsProper}(\mathbf{v}')$ & $\mathbf{v}' \subseteq \operatorname{int}\mathbf{v}$ **then**
8 $\qquad\texttt{Insert}(\mathcal{L},\mathbf{x})$
9 $\quad$**else if** $\texttt{IsProper}(\mathbf{v}')$ **then**
10 $\qquad$Refresh derivatives enclosures;
11 $\qquad\mathbf{s} \leftarrow -f(\hat{\mathbf{u}},\hat{\mathbf{v}}',\hat{\mathbf{x}}) - \mathbf{J}^u(\operatorname{dual}\mathbf{u}-\hat{\mathbf{u}}) - \mathbf{J}^v(\mathbf{v}'-\hat{\mathbf{v}}')$;
12 $\qquad\mathbf{x}' \leftarrow \hat{\mathbf{x}} + \Gamma^*\big(\mathbf{J}^x, \mathbf{s}, \mathbf{x}-\hat{\mathbf{x}}\big)$;
13 $\qquad$**if** $\texttt{IsProper}(\mathbf{x}')$ **then**
14 $\qquad\quad(\mathbf{x}'',\mathbf{x}''') \leftarrow \texttt{Bisect}(\mathbf{x}')$;
15 $\qquad\quad\texttt{Insert}(\mathcal{H},(\mathbf{x}'',\mathbf{u},\mathbf{v}'))$;
16 $\qquad\quad\texttt{Insert}(\mathcal{H},(\mathbf{x}''',\mathbf{u},\mathbf{v}'))$;
17 $\qquad$**end**
18 $\quad$**end**
19 **end**

of Sections 4.1 and 4.2, the algorithm is correct. The additional bisection of the box $\mathbf{u}$ is done by associating to a box $\mathbf{x}$ a list of boxes $(\mathbf{u}^{(k)},\mathbf{v}^{(k)})$, $k \in \mathcal{K}$, instead of only one box $(\mathbf{u},\mathbf{v})$. The corresponding semantics

$$\mathbf{x} \subseteq \mathbb{S}_f(\mathbf{u},\mathbf{v}) \iff (\forall k \in \mathcal{K}) \left(\mathbf{x} \subseteq \mathbb{S}_f(\mathbf{u}^{(k)},\mathbf{v}^{(k)})\right)$$

is then implemented without any difficulties.

5. EXPERIMENTATIONS

Our algorithm has been implemented in C/C++ and executed on a PentiumM 1,4Ghz with 512Mo memory. It is compared to a simpler algorithm which does not use the generalized HS operator to prune the box $\mathbf{x}$: this simpler algorithm is obtained by discarding lines 10–13 and 17 and changing $\mathbf{x}'$ into $\mathbf{x}$ in line 14. The results obtained using this simpler algorithm are written between parenthesis in the tables 1 and 2 when they are different. In the next two subsections, we wish to approximate

$$\mathbb{S} = \{x \in \mathbf{x} \mid (\forall u \in \mathbf{u})\,(\exists v \in \mathbf{v})\,(f(u,v,x) = 0)\}.$$

5.1 An academic example

We consider the function

$$f(u,v,x) = \begin{pmatrix} (v_1-u_3)^2 + (v_2-u_4)^2 - (x_1+u_1)^2 \\ (v_1-u_5)^2 + (v_2-u_6)^2 - (x_2+u_2)^2 \end{pmatrix},$$

so that $f(u,v,x) = 0$ means that the point v is at the intersection of two circles centered respectively at $(u_3,u_4)^T$ and $(u_5,u_6)^T$ and with respective radius x_1+u_1 and x_2+u_2. We will consider the following boxes: $\mathbf{u}_k = [\hat{\mathbf{u}}_k - \epsilon_k, \hat{\mathbf{u}}_k + \epsilon_k]$ with $\hat{\mathbf{u}}_5 = 10$ and $\hat{\mathbf{u}}_k = 0$ for $k \neq 5$. Furthermore $\mathbf{v} = ([3,7],[3,7])^T$ and $\mathbf{x} = ([4,10],[4,10])^T$. We define seven AE-solution sets depending on the uncertainties considered: $\mathbb{S}_i$, $i \in [0..6]$, is defined with $\epsilon_k = 1$ for $k \leq i$ and $\epsilon_k = 0$ for $k > i$. The results obtained using our algorithm with a bisection precision $\omega = 10^{-2}$ are shown in Table 1. The abbreviations "N.B.", "I.S." and "U.S." mean respectively Number of Bisections, final Inner Surface and final Unknown Surface. The symbol $*$ means *zero exactly* (so that $\mathbb{S}_4$, $\mathbb{S}_5$ and $\mathbb{S}_6$ are proved to be empty). Also, the left hand side graphic of Figure 1 shows the computed paving which approximates $\mathbb{S}_0$. These results show that the pruning power

	time (s)	N.B. (10^3)	I.S.	U.S.
$\mathbb{S}_0$	0.57(0.65)	6.8(10)	15	0.19(0.21)
$\mathbb{S}_1$	1.7(2.8)	15(30)	6.3	0.13(0.24)
$\mathbb{S}_2$	9.4(19)	61(136)	1.2	0.087(0.15)
$\mathbb{S}_3$	14(41)	83(237)	0.0038	0.0074(0.016)
$\mathbb{S}_4$	19(61)	117(396)	$*$	$*$
$\mathbb{S}_5$	27(120)	165(765)	$*$	$*$
$\mathbb{S}_6$	29(310)	181(1616)	$*$	$*$

Table 1: Computational results

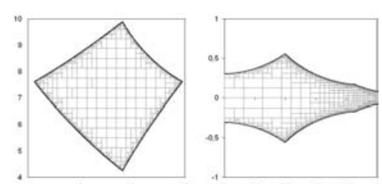

Figure 1: Paving for $\mathbb{S}_0$ (left) and $\mathbb{S}_7$ (right).

offered by generalized interval is important: although classical intervals are enough to ensure the convergence, the more uncertainties are important the more the generalized interval pruning accelerates the computations and reduces the final unknown volume.

5.2 Equilibrium of an aircraft

In [7], the equilibrium of an aircraft is studied thanks to the quantifier elimination ([3]). The moments acting on the aircraft are modeled by the following function:

$$
\begin{aligned}
f_1(u,v,x) = &-38x_2 - 170x_1x_2 + 148x_1^2x_2 + 4x_2^3 \\
&+v_1\big(-52 - 2x_1 + 114x_1^2 - 79x_1^3 + 7x_2^2 + 14x_1x_2^2\big) \\
&+v_3\big(14 - 10x_1 + 37x_1^2 - 48x_1^3 + 8x_1^4 - 13x_2^2 \\
&\quad -13x_1x_2^2 + 20x_1^2x_2^2 + 11x_2^4\big) \\
f_2(u,v,x) = &-12 - 125u_2 + u_2^2 + 6u_2^3 + 95x_1 - 21u_2x_1 \\
&+17u_2^2x_1 - 202x_1^2 + 81u_2x_1^2 + 139x_1^3 \\
f_3(u,v,x) = &\ 139x_2 - 112x_1x_2 - 388x_1^2x_2 + 215x_1^3x_2 \\
&-38x_2^3 + 185x_1x_2^3 \\
&+u_1(-11 + 35x_1 - 22x_1^2 + 5x_2^2 + 10x_1^3 - 17x_1x_2^2) \\
&+u_3(-44 + 3x_1 - 63x_1^2 + 34x_2^2 + 142x_1^3 \\
&\quad +63x_1x_2^2 - 54x_1^4 - 69x_1^2x_2^2 - 26x_2^4)
\end{aligned}
$$

In this function, x_k are the two angles of the airflow, v_k are the control-surface deflections and the parameters u_k are the fifteen constants appearing in the expression of f. Then $\mathbb{S}$ is the set of airflow angles for which some control-surface deflections exist such that the aircraft is equilibrated. In [7] the domains $\mathbf{x} = ([0,1],[-1,1])^T$ and $\mathbf{v} = ([-1,1],[-1,1],[-1,1])^T$ are used. We will study two situations: the first AE-solution set, called $\mathbb{S}_7$, is considered with no uncertainty, i.e. $\mathbf{u}$ is degenerated. The second AE-solution set, called $\mathbb{S}_8$, is considered with uncertainties $[-0.1, 0.1]$ added to each parameter. In this situation, we disabled the bisection in the space u which is of dimension fifty. The results obtained using our algorithm with a bisection precision $\omega = 10^{-3}$ are shown in Table 2. Also the right hand side of Figure 1 shows the computed paving which approximates $\mathbb{S}_7$. We can see that

	time (s)	N.B. (10^3)	I.S.	U.S.
$\mathbb{S}_7$	3.2(2.6)	11(13)	0.62	0.0040(0.0066)
$\mathbb{S}_8$	7.0(8.9)	26(57)	0.60	0.016(0.046)

Table 2: Computational results

pruning the box $\mathbf{x}$ both reduces the number of bisections and retracts the final unknown volume. In the meantime, it slows down the computations when no uncertainty is considered. However, when some uncertainties are considered, pruning the box $\mathbf{x}$ furthermore accelerates the computations.

6. RELATED WORK

The quantifier elimination ([3]) cannot solve directly the presented examples although a human simplification of the aircraft equilibrium problem (when no uncertainties are considered, i.e. $\mathbb{S}_7$) allowed to solve it in [7]. The paving computed by our algorithm is very similar to the solution obtained in [7]. Up to our knowledge, our algorithm is the only one that can compute some inner approximation of the other AE-solution sets without any preliminary formal simplification. When the AE-solution is empty, i.e. $\mathbb{S}_5$, $\mathbb{S}_6$ and $\mathbb{S}_7$, the solver RSolver, designed by Ratschan ([10]), should be able to prove this emptiness. However, tests have been done on the same machine and RSolver failed to prove the emptiness of any tested empty AE-solution sets (Rsolver was stopped after one hour of computations in each case).

7. CONCLUSION

The inner and outer approximations of AE-solution sets allow to model many practical situations and are actual subjects of research. In the cases where some existentially quantified parameters are shared between several equations, no numerical algorithm has yet been proposed to compute inner approximation of non-linear AE-solution sets.

We proposed a new branch and prune algorithm that is the first numerical algorithm able to compute both inner and outer approximations of such AE-solution sets. A new parametric Hansen-Sengupta operator has been used for both inner and outer approximations. Generalized intervals have been used to speed up the computations. And indeed the computations have been accelerated thanks to generalized intervals on all tested examples when some uncertainties are considered. Some promising experimentations have been conducted. In particular, on the empty AE-solution sets which have been tested, our algorithm was much more efficient than Ratschan's state of the art RSolver.

Forthcoming work will be to overcome some restrictions of the proposed algorithm. On one hand, the bisection of the space of existentially quantified variables was not yet implemented. On the other hand, some work has to be conducted to deal with AE-solution sets where there are more existentially quantified variables than the equations.

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