Quantifier Elimination versus Generalized Interval Evaluation: A Comparison on a Specific Class of Quantified Constraints

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Abstract

This paper presents and compares two methods for checking if a box is included inside the solution set of an equality constraint with existential quantification of its parameters. We focus on distance constraints, where each existentially quantified parameter has only one occurrence, because of their usefulness and their simplicity. The first method relies on a specific quantifier elimination based on geometric considerations whereas the second method relies on computations with generalized intervals—interval whose bounds are not constrained to be ordered. We show that on two dimension problems, the two methods yield equivalent results. However, when dealing with higher dimensions, generalized intervals are more efficient.

Keywords: Inner approximation, distance constraint, AE-solution set, quantifier elimination, generalized intervals, continuous domains.

1 Introduction

The interval theory ([13, 6]) was born in the 60's aiming rigorous computations with uncertain quantities. Interval constraint propagation ([1, 2]) is a widely used technique that allows one to reduce the domains of variables involved in a numerical constraint without losing any solution. When this technique is coupled with a bisection algorithm, an accurate reliable outer approximation of the solution set of a numerical Constraint Satisfaction Problem (CSP) can be achieved ([8]). However, when the solution set has a non-null (hyper)volume, such a branch and prune algorithm will bisect again and again the boxes included inside the solution set, leading to inefficient computations. This situation can be strongly improved using a test for detecting inner boxes so that boxes which are proved to lie inside the solution set will not be bisected any more. Furthermore, in addition to the speedup of computations, such inner boxes often have interesting interpretations.

There are different situations where the solution set of a CSP has a non-null volume, e.g. inequality constraints ([11]) or constraints with existentially quantified parameters, e.g. a constraint on variable $x \in \mathbb{R}$ like $(\exists a \in \mathbf{a})\,(c(a, x))$ where $\mathbf{a}$ is an interval ([7]). In this paper, we focus on quantified distance constraints where the variables are the coordinates of a point $x \in \mathbb{R}^n$. As existentially quantified parameters, we have the coordinates of another point $a \in \mathbb{R}^n$ and a distance $r \in \mathbb{R}$. Then, the distance constraint fixes the distance between a and x to be equal to r. The approximation of such constraints can be useful in many contexts, e.g. GPS localization or parallel robots modeling ([15, 12]).

We propose and compare two different methods for checking if a box is included inside the solution set of a distance equation with existentially quantified parameters. On one hand, the quantified distance constraint is changed to an equivalent non quantified disjunction/conjunction of constraints which can be checked using interval arithmetic. On the other hand, the Kaucher arithmetic of generalized intervals ([9, 4]), which represents a new formulation of the modal intervals theory ([14, 7]), allows one to verify the inclusion through a generalized interval evaluation of the constraint. These two tests for inner boxes are implemented in a branch and prune algorithm and experiments have been carried out on academic examples in order to compare them.

Notations Following [10], intervals are denoted by boldface letters. Integral intervals are denoted by $[m..n]$. Let $\mathcal{E} = \{e_1, ..., e_n\}$ be an ordered set of indices, the vector $(\mathbf{x}_{e_1}, ..., \mathbf{x}_{e_n})$ is denoted by $\mathbf{x}_{\mathcal{E}}$, so that $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is denoted by $\mathbf{x}_{[1..n]}$. If no confusion is possible, the usual notation $\mathbf{x}$ will be used in place of $\mathbf{x}_{[1..n]}$.

2 Problem statement

The Euclidean distance between the points $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ is defined by

$$f(a, x) = \sqrt{\Big(\sum_{k\in[1..n]} (x_k - a_k)^2 \Big)}$$

Given two n-dimensional boxes $\mathbf{x}$ and $\mathbf{a}$ and an interval $\mathbf{r}$, we are interested in the following quantified distance constraint

$$(\exists a \in \mathbf{a})\,(\exists r \in \mathbf{r})\,(f(a, x) = r) \qquad (1)$$

which is denoted by $c_{\mathbf{a},\mathbf{r}}(x)$. The set of $x \in \mathbb{R}^n$ which satisfies (1) is denoted by $\rho_{\mathbf{a},\mathbf{r}}$. It is shown in Figure 1: on the left hand side graphic, the center is known exactly ($\mathbf{a} = (1, 1)$) while the radius is known with uncertainty ($\mathbf{r} = [0.9, 1.1]$). On the right hand side graphic, all parameters are known with uncertainty ($\mathbf{a} = ([0.8, 1.2], [0.8, 1.2])$ and $\mathbf{r} = [0.9, 1.1]$), leading to a less intuitive graph. This paper aims to provide some sufficient conditions for the inclusion $\mathbf{x} \subseteq \rho_{\mathbf{a},\mathbf{r}}$.

It can be noted that a sufficient condition designed for one quantified distance constraint can also be used for a conjunction of quantified distance constraints $\bigwedge_{k\in[1..m]} c_{\mathbf{a}^{(k)},\mathbf{r}^{(k)}}(x)$,

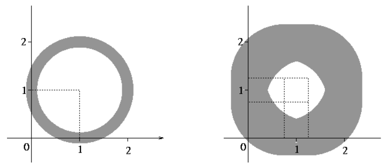

Figure 1: Quantified distance constraints.

where $\mathbf{a}^{(k)}$ are some n-dimensional boxes and $\mathbf{r}^{(k)}$ some intervals. Indeed, if existentially quantified parameters are not shared between different constraints, we have the following implication:

$$\bigwedge_{k\in[1..m]} \mathbf{x} \subseteq \rho_{\mathbf{a}^{(k)},\mathbf{r}^{(k)}} \quad \Longrightarrow \quad \mathbf{x} \subseteq \bigcap_{k\in[1..m]} \rho_{\mathbf{a}^{(k)},\mathbf{r}^{(k)}}$$

3 A specific quantifier elimination

The quantifier elimination (QE) consists in transforming a quantified constraint into an equivalent non quantified constraint. A general QE algorithm for polynomial constraints is available ([3]). However, its high complexity restricts its application to small problems. In the particular case of distance constraints, the implementation of QE proposed in Mathematica5.1 succeeds only in the 1-dimensional case. For higher dimensions, the calculus could not be ended before memory overflow on a Pentium IV 2Ghz with 512Mo of memory. In this section, we present a specific QE for the distance constraint $c_{\mathbf{a},\mathbf{r}}(x)$. The presentation is done in the two dimensional case. The three dimensional case can be treated in the same way. However, higher dimensions are still out of the scope of the proposed specific QE.

The typical graph of the constraint $c_{\mathbf{a},\mathbf{r}}(x)$ is shown in the left side of Figure 2, while its eight characteristic circles are represented on the right side. These circles are obtained using the bounds of the intervals $\mathbf{a}$ and $\mathbf{r}$ within the distance constraint. We constructed the graph of the left side graphic using the informations of the right side graphic, i.e. using only the bounds of the involved intervals.

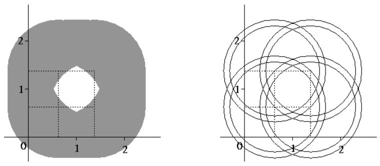

Figure 2: Quantified distance constraint together with its eight charateristic circles.

3.1 Decomposition of the quantified distance constraint

The specific QE proposed here relies on the decomposition of $c_{\mathbf{a},\mathbf{r}}(x)$ into two auxiliary constraints with convex graph. The graphs of these two auxiliary constraints are illustrated on Figure 3: let us call the left side constraint $c'_{\mathbf{a},\mathbf{r}}(x)$ and the right side one $c''_{\mathbf{a},\mathbf{r}}(x)$. Notice that the boundary of the right hand side graph is not included in the graph of $c''_{\mathbf{a},\mathbf{r}}(x)$ so that we clearly have

$$c_{\mathbf{a},\mathbf{r}}(x) \iff c'_{\mathbf{a},\mathbf{r}}(x) \wedge \neg c''_{\mathbf{a},\mathbf{r}}(x)$$

(see [5] for a proof in the general case taking into account some non typical situations). We now characterize these two auxiliary constraints using the bounds of the involved intervals.

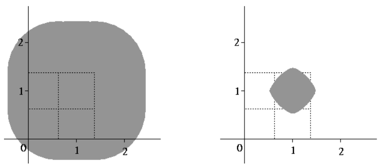

Figure 3: The two constraints $c'_{\mathbf{a},\mathbf{r}}(x)$ and $c''_{\mathbf{a},\mathbf{r}}(x)$ used in the reconstruction of $c_{\mathbf{a},\mathbf{r}}(x)$.

3.1.1 The constraint $c'_{\mathbf{a},\mathbf{r}}(x)$

The graph of the constraint $c'_{\mathbf{a},\mathbf{r}}(x)$ is built using the four exterior circles of Figure 3 and two boxes. Indeed, $c'_{\mathbf{a},\mathbf{r}}(x)$ is equivalent to the disjunction of the following six constraints:

1. $f(\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, x) \leq \overline{\mathbf{r}}$
2. $f(\overline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, x) \leq \overline{\mathbf{r}}$
3. $f(\underline{\mathbf{a}}_1, \overline{\mathbf{a}}_2, x) \leq \overline{\mathbf{r}}$
4. $f(\overline{\mathbf{a}}_1, \overline{\mathbf{a}}_2, x) \leq \overline{\mathbf{r}}$
5. $x \in ([\underline{\mathbf{a}}_1 - \overline{\mathbf{r}}, \overline{\mathbf{a}}_1 + \overline{\mathbf{r}}], \mathbf{a}_2)$
6. $x \in (\mathbf{a}_1, [\underline{\mathbf{a}}_2 - \overline{\mathbf{r}}, \overline{\mathbf{a}}_2 + \overline{\mathbf{r}}])$

This reconstruction of $c'_{\mathbf{a},\mathbf{r}}(x)$ is illustrated by Figure 4: the first four constraints represent four disks (left hand side graph of Figure 4). The graph of their disjunction is close to the graph of $c'_{\mathbf{a},\mathbf{r}}(x)$ but some gaps are present. The last two constraints fill the remaining gaps thanks to two boxes (right hand side of Figure 4).

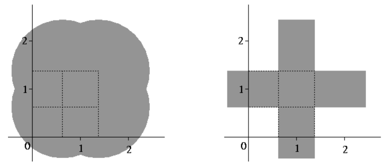

Figure 4: Reconstruction of the constraint $c'_{\mathbf{a},\mathbf{r}}(x)$ using four disks and two boxes.

3.1.2 The constraint $c''_{\mathbf{a},\mathbf{r}}(x)$

The graph of the constraint $c''_{\mathbf{a},\mathbf{r}}(x)$ is easily obtained by intersecting four open disks corresponding to the interior circles of Figure 2. The constraint $c''_{\mathbf{a},\mathbf{r}}(x)$ is equivalent to the conjunction of the four following constraints:

1. $f(\underline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, x) < \underline{\mathbf{r}}$
2. $f(\overline{\mathbf{a}}_1, \underline{\mathbf{a}}_2, x) < \underline{\mathbf{r}}$
3. $f(\underline{\mathbf{a}}_1, \overline{\mathbf{a}}_2, x) < \underline{\mathbf{r}}$
4. $f(\overline{\mathbf{a}}_1, \overline{\mathbf{a}}_2, x) < \underline{\mathbf{r}}$

Finally, $\neg c''_{\mathbf{a},\mathbf{r}}(x)$ is expressed as the disjunction of four (non-strict) inequalities.

3.2 Interval arithmetic

Now, we use classic interval arithmetic to evaluate the previously constructed expression for all x in a given box $\mathbf{x}$, in the following way:

- for constraints 5. and 6. of section 3.1.1, we have $\mathbf{x} \subseteq \tilde{\mathbf{a}} \Longrightarrow (\forall x \in \mathbf{x})\,(x \in \tilde{\mathbf{a}})$, where $\tilde{\mathbf{a}}$ is one of the two intervals involved in the constraints.
- for the other constraints, the natural extension of f is used (the real operations are replaced by their interval counterparts in the expression of f): we have $f(\tilde{a}, \mathbf{x}) \circ \tilde{r} \Longrightarrow (\forall x \in \mathbf{x})\,(f(\tilde{a}, x) \circ \tilde{r})$ where $\circ \in \{\leq, \geq\}$ and $\tilde{a}_1 \in \{\underline{\mathbf{a}}_1, \overline{\mathbf{a}}_1\}$,

$\tilde{a}_2 \in \{\underline{\mathbf{a}}_2, \overline{\mathbf{a}}_2\}$, and $\tilde{r} \in \{\underline{r}, \overline{r}\}$ are some bounds of the original intervals.

We now have a sufficient condition for $\mathbf{x} \subseteq \rho_{\mathbf{a},\mathbf{r}}$. But this is not a necessary condition: a box can satisfy $\mathbf{x} \subseteq \rho_{\mathbf{a},\mathbf{r}}$ while it does not satisfy any of the six constraints presented in Section 3.1.1. Such a box would intersect several graphs among the ones presented in Section 3.1.1 but would be included in none of them (this flaw will be called the *decomposition flaw* from now on). However, it can be proved that given a box satisfying $\mathbf{x} \subseteq \rho_{\mathbf{a},\mathbf{r}}$, the proposed sufficient condition will prove this inclusion after a finite number of midpoint bisections.

4 Generalized interval evaluation

In this section a sufficient condition is proposed for a n-dimensional box $\mathbf{x}$ to satisfy $\mathbf{x} \subseteq \rho_{\mathbf{a},\mathbf{r}}$. It is based on one evaluation of the expression of $f(x, a)$ using generalized intervals and their arithmetic. This technique was initially proposed in the modal intervals theory (see [14, 7]) and is now informally presented in a revisited way using generalized intervals (see [4] for a detailed presentation of the new formulation of the modal intervals theory). This new formulation allows one to understand the underlying mechanisms.

4.1 Generalized intervals and quantifiers

The intervals usually considered in the interval theory are closed, bounded and nonempty. These intervals are uniquely defined by two real numbers, called their bounds. The lower bound of an interval is of course lower or equal than its upper bound. Generalized intervals are defined relaxing the constraint that bounds have to be ordered, e.g. $[-1, 1]$ is a proper interval and $[1, -1]$ is an improper interval. So, related to a set of reals $\{x \in \mathbb{R} \mid u \leq x \leq v\}$, where $u, v \in \mathbb{R}$, one can consider two generalized intervals $[u, v]$ and $[v, u]$. It will be convenient to use the operations dual $[u, v] = [v, u]$ and pro $[u, v] = [\min\{u, v\}, \max\{u, v\}]$ (called proper projection) to change the proper/improper quality of a generalized interval keeping unchanged the underlying set of reals. The set of generalized intervals is denoted by $\mathbb{KR}$, the set of proper intervals by $\mathbb{IR}$ and the set of improper intervals by $\overline{\mathbb{IR}}$.

An inclusion is defined for generalized intervals by $\mathbf{x} \subseteq \mathbf{y} \iff \underline{\mathbf{y}} \leq \underline{\mathbf{x}} \wedge \overline{\mathbf{x}} \leq \overline{\mathbf{y}}$, e.g. $[-1, 1] \subseteq [-1.1, 1.1]$ (the inclusion corresponds to the inclusion between the underlying sets of reals), $[1.1, -1.1] \subseteq [1, -1]$ (the inclusion between the underlying sets of reals is reversed) and $[2, 0.9] \subseteq [-1, 1]$ (the underlying sets of reals have at least one common point).

Let us now consider a continuous function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, a generalized interval vector $\mathbf{x} \in \mathbb{KR}^n$ and a generalized interval $\mathbf{z} \in \mathbb{KR}$. We now define the $(g, \mathbf{x})$-interpretability of $\mathbf{z}$ as following: first, if $\mathbf{x}_1, \ldots, \mathbf{x}_n$, and $\mathbf{z}$ are proper then by definition $\mathbf{z}$ is $(g, \mathbf{x})$-interpretable if and only if

$$(\forall x \in \mathbf{x})\,(\exists z \in \mathbf{z})\,(g(x) = z)\,.$$

Therefore, when all involved intervals are proper, we obtain the interpretation of the classical interval theory. Second, if an improper interval is involved in place of a proper one, the related quantifier is changed in the quantified proposition to be satisfied, keeping the universal quantifiers in front of the quantified proposition. Also, when an improper interval is associated to a variable, the domain of this variable is the proper projection of the former. For example, if $\mathbf{x}_1, \mathbf{z} \in \mathbb{IR}$ (proper) and $\mathbf{x}_2 \in \overline{\mathbb{IR}}$ (improper) then by definition $\mathbf{z}$ is $(g, \mathbf{x})$-interpretable if and only if

$$(\forall x_1 \in \mathbf{x}_1)\,(\exists z \in \mathbf{z})\,(\exists x_2 \in \text{pro } \mathbf{x}_2)\,(g(x) = z)\,.$$

On the other hand, if $\mathbf{x}_1 \in \mathbb{IR}$ and $\mathbf{x}_2, \mathbf{z} \in \overline{\mathbb{IR}}$ then by definition $\mathbf{z}$ is $(g, \mathbf{x})$-interpretable if and only if

$$(\forall x_1 \in \mathbf{x}_1)\,(\forall z \in \text{pro } \mathbf{z})\,(\exists x_2 \in \text{pro } \mathbf{x}_2)\,(g(x) = z)\,.$$

Thanks to the definition of $(g, \mathbf{x})$-interpretable intervals, we are able to handle quantified propositions by only performing computations on generalized intervals. This will lead to efficient computations (see [4] for more details).

The definition of interpretable generalized intervals is stated, the next step is to construct such generalized intervals. This construction follows the construction of classical interval extensions: first the construction is done for simple functions (Subsection 4.2). This leads to some formal expressions of interpretable intervals in the cases of simple elementary functions like $+$, $-$, $\times$, $\div$, x^2, ... As in the context classical interval analysis, these expressions form a generalized interval arithmetic (that is proved to coincide with the Kaucher arithmetic). Then this generalized interval arithmetic is used to perform some generalized interval evaluation of the function (Subsection 4.3). Although it is not true in general, this evaluation is proved to compute interpretable generalized intervals when the expression used for the interval evaluation contains only one occurrence of each variable (see [4] for the specific treatment of expressions that contain multiple occurrences of variables). Therefore this generalized interval evaluation can be used for distance equations.

4.2 The Kaucher arithmetic

The Kaucher arithmetic extends the classical interval arithmetic to generalized intervals (see [9, 14, 4]). The Kaucher addition and subtraction have the same expressions than their classical counterparts: $\mathbf{x}+\mathbf{y} = [\underline{\mathbf{x}}+\underline{\mathbf{y}}, \overline{\mathbf{x}}+\overline{\mathbf{y}}]$ and $\mathbf{x}-\mathbf{y} = [\underline{\mathbf{x}}-\overline{\mathbf{y}}, \overline{\mathbf{x}}-\underline{\mathbf{y}}]$. Also, when only intervals with positive bounds are involved, the Kaucher multiplication has the following expression: $\mathbf{x}\times\mathbf{y} = [\underline{\mathbf{x}}\times\underline{\mathbf{y}}, \overline{\mathbf{x}}\times\overline{\mathbf{y}}]$. In the general case, the expressions of the Kaucher multiplication and division are more complicated. Although it was not introduced with this goal, the Kaucher operation $\mathbf{x}\circ\mathbf{y}$, where $\circ \in \{+,-,\times,/\}$, is proved to compute a $(\circ,\mathbf{x},\mathbf{y})$-interpretable generalized intervals (see [14, 4]). For example, $[1,2]+[10,4] = [11,6]$ is interpreted by

$$(\forall x \in [1,2])\,(\forall z \in [6,11])\,(\exists y \in [4,10])\;(x+y=z)$$

and $[1,2]\times[10,4] = [10,8]$ is interpreted by

$$(\forall x \in [1,2])\,(\forall z \in [8,10])\,(\exists y \in [4,10])\;(x\times y=z)$$

Also, univariate functions $f(x)$ like x^2 or $\sqrt{x}$ are extended to generalized intervals in the following way: the interval $f(\mathbf{x})$ satisfies $\mathrm{pro}\, f(\mathbf{x}) = \mathrm{range}\,(f, \mathrm{pro}\,\mathbf{x})$ and has the same proper/improper quality than $\mathbf{x}$, e.g. $[2,3]^2 = [4,9]$ is interpreted by $(\forall x \in [2,3])\,(\exists z \in [4,9])\,\big(x^2=z\big)$ and $[3,2]^2 = [9,4]$ is interpreted by $(\forall z \in [4,9])\,(\exists x \in [2,3])\,\big(x^2=z\big)$.

Now, as we can compute interpretable intervals for elementary functions, we are in position to provide interpretable intervals for more realistic functions compounded of elementary functions.

4.3 Generalized evaluation of an expression

Let us illustrate on an example the generalized interval evaluation of an expression. Consider the function $g(x,a,u) = (x-a)^2+u$ and the generalized intervals $\mathbf{a} = [4,2]$, $\mathbf{x} = [0,1]$ and $\mathbf{u} = [2,3]$. The interval $\mathbf{z}$ is obtained by evaluating the expression of g (the intervals $\mathbf{t}$ and $\mathbf{s}$ are intermediate intervals): $\mathbf{t} = \mathbf{x}-\mathbf{a} = [-2,-3]$, $\mathbf{s} = \mathbf{t}^2 = [9,4]$ and $\mathbf{z} = \mathbf{s}+\mathbf{u} = [11,7]$. These computations are interpreted by the following quantified propositions: $\mathbf{t} = \mathbf{x}-\mathbf{a} = [-2,-3]$ is interpreted by

$$(\forall x \in \mathbf{x})\,(\forall t \in \mathrm{pro}\,\mathbf{t})\,(\exists a \in \mathrm{pro}\,\mathbf{a})\;(x-a=t)\,. \tag{2}$$

Also $\mathbf{s} = \mathbf{t}^2 = [9,4]$ is interpreted by

$$(\forall s \in \mathrm{pro}\,\mathbf{s})\,(\exists t \in \mathrm{pro}\,\mathbf{t})\;\big(t^2=s\big)\,. \tag{3}$$

Finally $\mathbf{z} = \mathbf{s}+\mathbf{u} = [11,7]$ is interpreted by

$$(\forall u \in \mathbf{u})\,(\forall z \in \mathrm{pro}\,\mathbf{z})\,(\exists s \in \mathrm{pro}\,\mathbf{s})\;(s+u=z)\,. \tag{4}$$

It is easy to see that the quantified propositions (2) and (3) imply the following one:

$$(\forall x \in \mathbf{x})\,(\forall s \in \mathrm{pro}\,\mathbf{s})\,(\exists a \in \mathrm{pro}\,\mathbf{a})\;\big((x-a)^2=s\big)\,. \tag{5}$$

In the same way, the quantified propositions (4) and (5) imply the following one:

$$(\forall x \in \mathbf{x})\,(\forall u \in \mathbf{u})\,(\forall z \in \text{pro } \mathbf{z})\,(\exists a \in \text{pro } \mathbf{a})\quad (g(x,a,u) = z)\,.$$

Therefore, the interval $\mathbf{z}$ is $(g, \mathbf{x}, \mathbf{a}, \mathbf{u})$-interpretable.

The presented argumentation is easily generalized to any expression containing only one occurrence of each variable and any generalized interval arguments, and therefore to quantified distance constraints of arbitrary dimension. As a consequence, the generalized interval evaluation $f(\text{dual } \mathbf{a}, \mathbf{x})$ yields a $(f, \text{dual } \mathbf{a}, \mathbf{x})$-interpretable interval. Furthermore, thanks to the properties of the generalized intervals inclusion (see [4]), if $\mathbf{r}$ satisfies $f(\text{dual } \mathbf{a}, \mathbf{x}) \subseteq \mathbf{r}$ then $\mathbf{r}$ is also $(f, \text{dual } \mathbf{a}, \mathbf{x})$-interpretable, that is

$$(\forall x \in \mathbf{x})\,(\exists a \in \mathbf{a})\,(\exists r \in \mathbf{r})\,(f(a,x) = r)$$

is true. Finally, the inclusion $f(\text{dual } \mathbf{a}, \mathbf{x}) \subseteq \mathbf{r}$ is a sufficient condition for $\mathbf{x} \subseteq \rho_{\mathbf{a},\mathbf{r}}$. This condition is not necessary in general, e.g. $\mathbf{a} = ([-2,2],[-2,2])$ and $\mathbf{r} = [1,1]$ so that $\mathbf{x} = ([-2,2],[-2,2])$ is an inner box which does not satisfy $f(\text{dual } \mathbf{a}, \mathbf{x}) \subseteq \mathbf{r}$ (in this case, the specific QE presented in Section 3 succeeds in proving the inclusion). However, it can be proved that the sufficient condition based on generalized interval evaluation is furthermore necessary provided that $\mathbf{x} \cap \mathbf{a} = \emptyset$. It is likely to be satisfied for inner boxes $\mathbf{x}$ in some realistic situations.

5 Comparison of the two methods

Some academic examples were selected in order to compare both approaches for checking inner boxes in a CSP involving only quantified distance constraints. Problem 1 and Problem 2 are in a two dimensional space, while Problem 3 is in a three dimensional space. The first problem is composed of a single constraint $c_{\mathbf{a},\mathbf{r}}(x)$, while the second and third problem are composed of three constraints $c_{\mathbf{a}^{(1)},\mathbf{r}^{(1)}}(x)$, $c_{\mathbf{a}^{(2)},\mathbf{r}^{(2)}}(x)$, and $c_{\mathbf{a}^{(3)},\mathbf{r}^{(3)}}(x)$. All problems have uncertainties. Table 1 shows the description of each one.

Table 1: Some academic examples.

$\mathbf{P_1}$ (2D, one equation)	
$\mathbf{x}$	$= ([-100,100],[-100,100])$
$\mathbf{a}$	$= ([-0.5,0.5],[-0.5,1.3])$
$\mathbf{r}$	$= [1.3,1.6]$
$\mathbf{P_2}$ (2D, three equations)	
$\mathbf{x}$	$= ([-100,100],[-100,100])$
$\mathbf{a}^{(1)}$	$= (0,0)$
$\mathbf{r}^{(1)}$	$= [2,2.25]$
$\mathbf{a}^{(2)}$	$= ([3,3.5],0)$
$\mathbf{r}^{(2)}$	$= [2.95,3.05]$
$\mathbf{a}^{(3)}$	$= ([-2.5,-2.25],2)$
$\mathbf{r}^{(3)}$	$= [3.25,3.5]$
$\mathbf{P_3}$ (3D, three equations)	
$\mathbf{x}$	$= ([0,100],[-100,100],[0,100])$
$\mathbf{a}^{(1)}$	$= ([-0.1,0.1],[-0.1,0.1],[-0.1,0.1])$
$\mathbf{r}^{(1)}$	$= [4,5]$
$\mathbf{a}^{(2)}$	$= ([4.9,5.1],[-0.1,0.1],[-0.1,0.1])$
$\mathbf{r}^{(2)}$	$= [3,4]$
$\mathbf{a}^{(3)}$	$= ([1.8,2.2],[3.95,4.05],[0.8,1.2])$
$\mathbf{r}^{(3)}$	$= [4,5]$

A branch and prune algorithm combining 2B-consistency and bisection techniques was used for solving each problem. The inner box checking was applied each time the consistency algorithm failed in reducing the space.

Table 2 shows the computational results[1] of the experiments, using the specific quantifier elimination (SQE) and the generalized interval evaluation (GIE). Rows *Box* and *Ibox* show the total number of boxes and the number of inner boxes found, respectively. Row *Vol* shows the total volume of the boxes, while row *Ivol* shows the volume of the inner boxes. Row *Time* shows the running time in seconds. Some experiments have been conducted without using any inner box checking, but Problem 1 led to swap memory overflow (1.6Go) before reaching the expected precision.

First of all, it is clear that the use of inner tests drastically reduces the computation times in all situations.

[1]Obtained on a Pentium IV 2GHz with 256Mb of RAM and 1,5Gb of swap memory, running IcosAlias v0.2b (http://www-sop.inria.fr/coprin/gchabert/icosalias.html).

Table 2: Computational results.

	No Test	SQE	GIE
$\mathbf{P_1}$			
Box	$> 10^7$	64877	64877
Ibox	–	33225	33225
Vol	–	18.50312	18.50312
Ivol	–	18.49187	18.49187
Time	–	4.63	4.08
$\mathbf{P_2}$			
Box	451655	5481	5481
Ibox	–	2550	2550
Vol	0.21236	0.21236	0.21236
Ivol	–	0.21103	0.21103
Time	36,08	0.53	0.43
$\mathbf{P_3}$			
Box	7717507	503059	501795
Ibox	–	137900	137799
Vol	2.83133	2.83133	2.83133
Ivol	–	2.72203	2.72254
Time	803.63	87.49	58.38

On Problem 1 and Problem 2 (Figure 5 and Figure 6), the two methods for inner box checking are optimal and compute exactly the same approximations: on one hand, the bisection is performed in such a way that the *decomposition flaw* (section 3.2) of the SQE is not met. On the other hand, we have $\mathbf{x} \cap \mathbf{a} = \emptyset$ for all inner boxes, so that the GIE is optimal. The running time using the GIE is always slightly lower than using the SQE because the former computes only one evaluation of the constraint.

On Problem 3 (Figure 7), the two tests compute different approximations: the total volumes are equal with both methods but the inner volume provided by the GIE is greater, with a lower number of inner boxes. While the GIE is still optimal (because $\mathbf{x} \cap \mathbf{a} = \emptyset$), the *decomposition flaw* is now met (in dimension 3 the decomposition used for the SQE is more complicated so the *decomposition flaw* is more likely to be met). As a consequence, the speedup of GIE is more sensitive on this example.

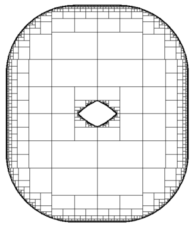

Figure 5: Solution for Problem 1.

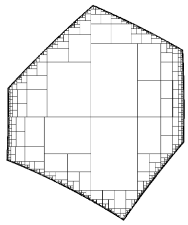

Figure 6: Solution for Problem 2.

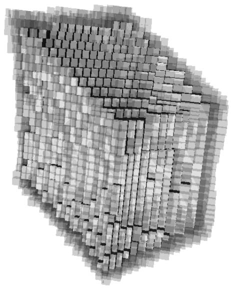

Figure 7: Solution for Problem 3.

6 Conclusion

Equality constraints with existentially quantified parameters, i.e. constraints like $(\exists a \in \mathbf{a})\,(f(a,x)=0)$, generally have a non-null volume solution set. Therefore, any bisection algorithm dedicated to the approximation of their solution set should incorporate a test for checking if a box is included inside the solution set, unless it will spend most of the time bisecting again and again boxes included in the solution set. Focusing on the simple but useful example of quantified distance constraints, we proposed two tests: on one hand, thanks to geometric considerations, the quantified distance constraint has been changed to a non-quantified constraint. On the other hand, we presented with a new point of view a test which was initially proposed by the modal intervals theory. This new formulation of the modal intervals theory allows one to understand the underlying mechanisms.

Some experiments have been conducted on academic examples of conjunctions of quantified distance constraints. Although both methods are very different, they yield very similar results about both computation times and description of the solution set (with a slight advantage for the test based on generalized intervals). Moreover, the test based on generalized interval evaluation presents two advantages: first, it is much simpler to implement. Second, it can be easily extended to a quantified distance constraint in an arbitrary dimensional space, where the proposed specific quantifier elimination fails.

As forthcoming work, a new inner test combining the two presented tests will be studied aiming to obtain an optimal test in all situations. Also, we will apply some inner test in the context of parallel robots study, taking into account the uncertainties on the geometric parameters.

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